The histograms of forces is a method to compute the relative position between objects. The types of objects that this method can handle are constrained to the following. Let $A$ be a two-dimensional, crisp or fuzzy object that is a nonempty bounded set of points, equal to its interior closure, such that, for any $\theta \in \mathbb{R}$ and $v \in \mathbb{R}$, $A \cap \Delta(v)$ is the intersection of a finite number of mutually disjoint segments. Here, $\Delta(v)$ is the oriented line whose frame is defined by the vector $i_0$ and the point of coordinates $(0, v)$, where $i_0$ is the image of a vector $i$ through a $\theta$-angle rotation. The set $A \cap \Delta(v) \neq \emptyset$, denoted henceforth as $A_\theta(v)$, is a longitudinal section of $A$ and is an object decomposition that capacitates robust spatial processing.

Consider now a couple of objects, $(A, B)$, that both satisfy the properties of $A$. For an arbitrary angle $\theta$, we would like to assess the weight of the proposition “$A$ is in direction $\theta$ of $B$.” To numerically express the support of this statement, a function $F_r : T \rightarrow \mathbb{R}^+$, where $T$ is defined as a set of triples, $T = \{(\theta, A_\theta(v), B_\theta(v))\} \cap \mathbb{R}^2$, and $r \in \mathbb{R}^+$, can be defined that operates upon collinear longitudinal sections of $A$ and $B$. However, directly processing sets of longitudinal sections is not a trivial task. As such, the function $F_r$ can be realized by further fragmenting $A$ and $B$ from longitudinal sections, $A_\theta(v)$ and $B_\theta(v)$, to aligned segments, $I \in A_\theta(v)$ and $J \in B_\theta(v)$, to pairs of points, $M \in I$ and $N \in J$. For each of these data types, it becomes necessary to define the aggregation operations that build $F_r$.

Since a longitudinal section is just a union of segments, and since a segment is just a union of points, it is prudent to first consider a function $\phi_r$ to handle points. Given two crisp objects, $A$ and $B$, we define two arbitrary points, $M$ and $N$, and $\phi_r(x_M - x_N)$, where $x_M$ and $x_N$ refer to the respective abscissas of $M$ and $N$ on $\Delta(v)$, is the weight of the argument. Provided that $x_M - x_N > 0$, there is some weight that supports the proposition that $M$ is in the direction $\theta$ of $N$. Conversely, if $x_M - x_N < 0$, there is no weight that supports the proposition that $M$ is in the direction $\theta$ of $N$. For any $r \in \mathbb{R}^+$, we can define a particular function instantiation, $\phi_r : \mathbb{R} \rightarrow \mathbb{R}^+$, null on $\mathbb{R}^-$ and continuous on $\mathbb{R}^+$, as $\phi_r = 1/d_{MN}$, where $d_{MN}$ represents the distance between points $M$ and $N$, (cf. fig. 1(a)). This realization for $r = 0$ corresponds to the constant force exerted by one object on another, while the same realization for $r = 2$ corresponds to the scalar resultant of the elementary forces of gravity. Building upon the mapping function for points, line segments can be handled via a function $f_r : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. For any number of $\theta$ values, there are an infinite number of couples, $(I, J)$, of aligned segments such that $(\theta, I, J) \subseteq T$. Considering only a single instance, there exists a single real number, $v$, whose oriented straight line, $\Delta(v)$, includes $I$ and $J$. Let the coordinates, relative to the frame associated with $\Delta(v)$, at the ends of segments $I$ and $J$ be $a_i^g$, $b_i^g$, $a_j^g$, $b_j^g$, where $a_i^g \leq b_i^g$ and $a_j^g \leq b_j^g$, and $d = b_i^g - a_i^g$, $d = b_j^g - a_j^g$, $d_a = a_i^g - b_i^g$, $d_j = a_j^g - b_j^g$ (cf. fig. 1(b)). Given the same two crisp objects, $A$ and $B$, as above, we consider that $(I, J)$ are an argument to support the proposition “$A$ is in direction $\theta$ of $B$.” Thus, $f_r(d_I, d_{IJ}^g, d_J)$ represents the weight of this argument, and only depends on the lengths of $I$ and $J$ and on the relative positions of these segments on $\Delta(v)$. Consequently, the values of $f_r(d_I, d_{IJ}^g, d_J)$ can be estimated by...
summing up the weights \( \phi_r(x_M - x_N) \) of the \((M,N)\) arguments, where \((M,N)\) fully describe \((I,J)\):

\[
f_r(d_I, d_{ij}, d_J) = \int_{a_I}^{b_I} \left( \int_{a_J}^{b_J} \phi_r(u-v) \, dv \right) \, du = \int_{d_i}^{d_i + d_{ij}} \left( \int_{0}^{d_j} \phi_r(u-v) \, dv \right) \, du.
\]

(1)

Furthering the mapping function for line segments, couples of longitudinal sections can be handled via the function \( F_r : T \to \mathbb{R}_+ \). Given the same two objects, \( A \) and \( B \), as above, there exists one set \( \{I_i\}_{i=1,...,n} \) of mutually disjoint segments, and only one such that \( A_0(v) = \bigcup_{i=1,...,n} I_i \). Similarly, there exists one set \( \{J_j\}_{j=1,...,m} \) of mutually disjoint segments, and only one such that \( B_0(v) = \bigcup_{j=1,...,m} J_j \) (cf. fig. 1(e)). The weight of the argument \((A_0(v), B_0(v))\) can be estimated by summing the weights \( f_r(d_I, d_{ij}, d_J) \):

\[
F_r(\theta, A_0(v), B_0(v)) = \sum_{i=1,...,n; j=1,...,m} f_r(d_I, d_{ij}, d_J)
\]

(2)

of the \((I,J)\) arguments, where \( I \) and \( J \) describe \( \{I_i\}_{i=1,...,n} \) and \( \{J_j\}_{j=1,...,m} \), respectively.

Finally, for any function \( F_r : T \to \mathbb{R}_+ \) and for any couple of crisp objects \((A, B)\), there exists a function \( F_r^{AB} \) that represents the total weight of the original proposition. “A is in direction \( \theta \) of \( B\),” for a single value of \( \theta \):

\[
F_r^{AB}(\theta) = \int_{-\infty}^{\infty} F_r(\theta, A_0(v), B_0(v)) \, dv.
\]

(3)

As with the other functions, an outline has been provided in fig. 1(d). In the case that \( A \) and \( B \) are fuzzy objects, and the total number of unique graylevel values of \( A \) and \( B \) is \( g \), then (3) may be re-written as either:

\[
F_r^{AB}(\theta) = \int_{-\infty}^{\infty} \sum_{i=1}^{g} (\alpha_i - \alpha_{i+1}) F_r(\theta, A_0(v)^{\alpha_i}, B_0(v)^{\alpha_i}) \, dv
\]

(4)

or

\[
F_r^{AB}(\theta) = \int_{-\infty}^{\infty} \sum_{i=1}^{g} \sum_{j=1}^{g} (\alpha_i - \alpha_{i+1})(\alpha_j - \alpha_{j+1}) F_r(\theta, A_0(v)^{\alpha_i}, B_0(v)^{\alpha_j}) \, dv.
\]

(5)

Above, \( A^{\alpha_i} \) denotes an \( \alpha \)-cut for the current value of \( \alpha_i \). When multiple values of \( \theta \in [-\pi, \pi] \) are considered by \( F_r^{AB} \), the result is a force histogram. Examples of force histograms for the objects in fig. 1 are shown in fig. 2.

For more information, please consult: